Problem Set 6 – Relativity (`GOI36A`)

1 Killing Vectors on S^2

For this problem, we will need the non-zero Christoffel symbols

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta \ , \quad \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta \ . \tag{1.1}$$

Also, note that we use the convention

$$V = V^{\mu}\partial_{\mu} \ , \tag{1.2}$$

for all vectors below.

1.) The metric tensor is independent of the coordinate ϕ, which means that coordinate changes that are rotations of the form $\phi \to \phi + c$, for some constant c, leave the metric invariant. This is an isometry of S^2, and intuitively the vector $A = \partial_{\phi}$ should be a Killing vector that generates this isometry.

   Let's prove this. First, we note that the Killing vector equation $\nabla_{(\mu}K_{\nu)} = 0$ for a general Killing vector K on S^2 has the following three independent components labelled by (μ, ν):

$$\begin{aligned}
(\theta,\theta): \quad 0 &= \nabla_{\theta}K_{\theta} \\
&= \partial_{\theta}K_{\theta} - \Gamma^{\mu}_{\theta\theta}K_{\mu} \\
&= \partial_{\theta}K_{\theta} \ . \\
(\phi,\phi): \quad 0 &= \nabla_{\phi}K_{\phi} \\
&= \partial_{\phi}K_{\phi} - \Gamma^{\mu}_{\phi\phi}K_{\mu} \\
&= \partial_{\phi}K_{\phi} - \Gamma^{\theta}_{\phi\phi}K_{\theta} \\
&= \partial_{\phi}K_{\phi} + \sin\theta\cos\theta\, K_{\theta} \ . \\
(\theta,\phi): \quad 0 &= \nabla_{\theta}K_{\phi} + \nabla_{\phi}K_{\theta} \\
&= \partial_{\theta}K_{\phi} + \partial_{\phi}K_{\theta} - 2\Gamma^{\mu}_{\theta\phi}K_{\mu} \\
&= \partial_{\theta}K_{\phi} + \partial_{\phi}K_{\theta} - 2\Gamma^{\phi}_{\theta\phi}K_{\phi} \\
&= \partial_{\theta}K_{\phi} + \partial_{\phi}K_{\theta} - 2\cot\theta K_{\phi} \ .
\end{aligned} \tag{1.3}$$

   If we express our coordinates by $x^{\mu} = (\theta, \phi)$, then the vector A has components $A^{\mu} = (0, 1)$, and so $A_{\mu} = g_{\mu\nu}A^{\nu} = (0, \sin^2\theta)$. Since $A_{\theta} = 0$ and $\partial_{\phi}A_{\phi} = 0$, the (θ,θ) and (ϕ,ϕ) Killing vector equation components are trivially satisfied. The (θ,ϕ) component is as follows:

$$\begin{aligned}
0 &\stackrel{!}{=} \partial_{\theta}A_{\phi} + \partial_{\phi}A_{\theta} - 2\cot\theta A_{\phi} \\
&= \partial_{\theta}(\sin^2\theta) + 0 - 2\cot\theta(\sin^2\theta) \\
&= 2\sin\theta\cos\theta - 2\sin\theta\cos\theta \\
&= 0 \ ,
\end{aligned} \tag{1.4}$$

   and thus A is a Killing vector.

2.) $C^\mu = (-\sin\phi, -\cot\theta\cos\phi)$, and so $C_\mu = (-\sin\phi, -\sin\theta\cos\theta\cos\phi)$. The three Killing vector equation components are as follows:

$$\begin{aligned}
(\theta,\theta): \quad 0 &\overset{!}{=} \partial_\theta C_\theta \\
&= \partial_\theta(-\sin\phi) = 0 \ . \\
(\phi,\phi): \quad 0 &\overset{!}{=} \partial_\phi C_\phi + \sin\theta\cos\theta C_\theta \\
&= \partial_\phi(-\sin\theta\cos\theta\cos\phi) + \sin\theta\cos\theta(-\sin\phi) \\
&= \sin\theta\cos\theta\sin\phi - \sin\theta\cos\theta\sin\phi = 0 \ . \\
(\theta,\phi): \quad 0 &\overset{!}{=} \partial_\theta C_\phi + \partial_\phi C_\theta - 2\cot\theta C_\phi \\
&= \partial_\theta(-\sin\theta\cos\theta\cos\phi) + \partial_\phi(-\sin\phi) - 2\cot\theta(-\sin\theta\cos\theta\cos\phi) \\
&= (-\cos^2\theta + \sin^2\theta)\cos\phi - \cos\phi + 2\cos^2\theta\cos\phi \\
&= (-\cos^2\theta + \sin^2\theta - 1 + 2\cos^2\theta)\cos\phi \\
&= (\cos^2\theta + \sin^2\theta - 1)\cos\phi \\
&= (1-1)\cos\phi = 0 \ .
\end{aligned} \tag{1.5}$$

They are all satisfied, and thus C is a Killing vector.

3.) First, we compute that

$$\begin{aligned}
CA &= (-\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi)(\partial_\phi) \\
&= -\sin\phi\,\partial_\theta\partial_\phi - \cot\theta\cos\phi\,\partial_\phi^2 \ .
\end{aligned} \tag{1.6}$$

The opposite is a little more involved, since A can act on the components of C directly:

$$\begin{aligned}
AC &= (\partial_\phi)(-\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi) \\
&= -\cos\phi\,\partial_\theta + \cot\theta\sin\phi\,\partial_\phi - \sin\phi\,\partial_\theta\partial_\phi - \cot\theta\cos\phi\,\partial_\phi^2 \ .
\end{aligned} \tag{1.7}$$

Notice that the terms in (1.6) and (1.7) that have two partial derivatives are the same, and so they will cancel when we take the difference. (You should be able to see that this is true in general when computing these commutators, since partial derivatives commute, and so in the future we will simply suppress these terms and only keep the terms where one free derivative is left.) The commutator is thus simply:

$$B = [C, A] = CA - AC = \cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi \ . \tag{1.8}$$

In components, $B_\mu = (\cos\phi, -\sin\theta\cos\theta\sin\phi)$. One can easily check that this satisfies the Killing vector equation:

$$\begin{aligned}
(\theta,\theta): \quad 0 &\overset{!}{=} \partial_\theta B_\theta \\
&= \partial_\theta(\cos\phi) = 0 \ . \\
(\phi,\phi): \quad 0 &\overset{!}{=} \partial_\phi B_\phi + \sin\theta\cos\theta B_\theta \\
&= \partial_\phi(-\sin\theta\cos\theta\sin\phi) + \sin\theta\cos\theta(\cos\phi) \\
&= -\sin\theta\cos\theta\cos\phi + \sin\theta\cos\theta\cos\phi = 0 \ . \\
(\theta,\phi): \quad 0 &\overset{!}{=} \partial_\theta B_\phi + \partial_\phi B_\theta - 2\cot\theta B_\phi \\
&= \partial_\theta(-\sin\theta\cos\theta\sin\phi) + \partial_\phi(\cos\phi) - 2\cot\theta(-\sin\theta\cos\theta\sin\phi) \\
&= (-\cos^2\theta + \sin^2\theta)\sin\phi - \sin\phi + 2\cos^2\theta\sin\phi \\
&= (-\cos^2\theta + \sin^2\theta - 1 + 2\cos^2\theta)\sin\phi \\
&= (\cos^2\theta + \sin^2\theta - 1)\sin\phi \\
&= (1-1)\sin\phi = 0 \ .
\end{aligned} \tag{1.9}$$

4.) We've already shown that $[C, A] = B$. Now, let's compute $[A, B]$:

$$\begin{aligned}
[A, B] &= \partial_\phi(\cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi) - (\cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi)\partial_\phi \\
&= (\partial_\phi\cos\phi)\partial_\theta - \cot\theta(\partial_\phi\sin\phi)\partial_\phi \\
&= -\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi \\
&= C\ .
\end{aligned} \tag{1.10}$$

Now, for $[B, C]$:

$$\begin{aligned}
[B, C] &= (\cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi)(-\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi) \\
&\quad - (-\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi)(\cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi) \\
&= \cos\phi\,\partial_\theta(-\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi) \\
&\quad - \cot\theta\sin\phi\,\partial_\phi(-\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi) \\
&\quad + \sin\phi\,\partial_\theta(\cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi) \\
&\quad + \cot\theta\cos\phi\,\partial_\phi(\cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi) \\
&= \cos\phi\,(\csc^2\theta\cos\phi\,\partial_\phi) \\
&\quad - \cot\theta\sin\phi\,(-\cos\phi\,\partial_\theta + \cot\theta\sin\phi\,\partial_\phi) \\
&\quad + \sin\phi\,(\csc^2\theta\sin\phi\,\partial_\phi) \\
&\quad + \cot\theta\cos\phi\,(-\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi) \\
&= (\csc^2\theta\cos^2\phi + \csc^2\theta\sin^2\phi - \cot^2\theta\sin^2\phi - \cot^2\theta\cos^2\phi)\partial_\phi \\
&\quad + (\cot\theta\sin\phi\cos\phi\partial_\theta - \cot\theta\sin\phi\cos\phi)\partial_\theta \\
&= (\csc^2\theta - \cot^2\theta)\partial_\phi \\
&= \partial_\phi \\
&= A\ .
\end{aligned} \tag{1.11}$$

Thus, we find that the Killing vectors satisfy the $SO(3)$ algebra. The group $SO(3)$ describes rotations in 3D; the three generators of the group are rotations in the $x - y$ plane, the $y - z$ plane, and the $x - z$ plane. The Killing vectors we find here are precisely these generators. S^2 should be invariant under such rotations, i.e. they are isometries of the spacetime, and so it is only natural for the isometry group of S^2 to be $SO(3)$.

2 Various Fun Things to Prove about Killing Vectors

5.) The crux of this problem is that the covariant derivative ∇_μ is a linear operator, and so if we have two killing vectors K_1 and K_2, and we take a linear combination of the form $K_3 = \alpha K_1 + \beta K_2$ for some constants α and β, then K_3 satisfies the Killing vector equation:

$$\begin{aligned}
\nabla_\mu K_{3\nu} + \nabla_\nu K_{3\mu} &= \nabla_\mu(\alpha K_{1\nu} + \beta K_{2\nu}) + \nabla_\nu(\alpha K_{1\mu} + \beta K_{2\mu}) \\
&= \alpha\nabla_\mu K_{1\nu} + \beta\nabla_\mu K_{2\nu} + \alpha\nabla_\nu K_{1\mu} + \beta\nabla_\nu K_{2\mu} \\
&= \alpha(\nabla_\mu K_{1\nu} + \nabla_\nu K_{1\mu}) + \beta(\nabla_\mu K_{2\nu} + \nabla_\nu K_{2\mu}) \\
&= \alpha\cdot 0 + \beta\cdot 0 \\
&= 0\ .
\end{aligned} \tag{2.1}$$

6.) **Bonus:** Let K and L be two Killing vectors, and define M to be their commutator:

$$\begin{aligned} M &= [K, L] = [K^\mu \partial_\mu, L^\nu \partial_\nu] \\ &= K^\mu \partial_\mu L^\nu \partial_\nu - L^\nu \partial_\nu K^\mu \partial_\mu \\ &= K^\mu (\partial_\mu L^\nu) \partial_\nu - L^\nu (\partial_\nu K^\mu) \partial_\mu \\ &= \left(K^\nu (\partial_\nu L^\mu) - L^\nu (\partial_\nu K^\mu)\right) \partial_\mu \ . \\ \implies M^\mu &= K^\nu (\partial_\nu L^\mu) - L^\nu (\partial_\nu K^\mu) \ . \end{aligned} \tag{2.2}$$

Now, note that $\partial_\nu L^\mu = \nabla_\nu L^\mu - \Gamma^\mu_{\nu\rho} L^\rho$, and thus

$$\begin{aligned} M^\mu &= K^\nu (\nabla_\nu L^\mu - \Gamma^\mu_{\nu\rho} L^\rho) - L^\nu (\nabla_\nu K^\mu - \Gamma^\mu_{\nu\rho} K^\rho) \\ &= K^\nu (\nabla_\nu L^\mu) - L^\nu (\nabla_\nu K^\mu) + \Gamma^\mu_{\nu\rho} (L^\nu K^\rho - K^\nu L^\rho) \ . \end{aligned} \tag{2.3}$$

The Christoffel symbols $\Gamma^\mu_{\nu\rho}$ are symmetric in their lower indices, while $L^\nu K^\rho - K^\nu L^\rho$ is antisymmetric in ν and ρ, and thus their contraction is zero. We are left with

$$M^\mu = K^\nu (\nabla_\nu L^\mu) - L^\nu (\nabla_\nu K^\mu) \ , \tag{2.4}$$

and thus

$$M_\mu = K^\nu \nabla_\nu L_\mu - L^\nu \nabla_\nu K_\mu \ . \tag{2.5}$$

We now need to prove that $\nabla_{(\mu} M_{\nu)}$ vanishes. To do this, we first expand it explicitly:

$$\begin{aligned} \nabla_\mu M_\nu + \nabla_\nu M_\mu &= \nabla_\mu (K^\rho \nabla_\rho L_\nu - L^\rho \nabla_\rho K_\nu) + \nabla_\nu (K^\rho \nabla_\rho L_\mu - L^\rho \nabla_\rho K_\mu) \\ &= \underbrace{\nabla_\mu K^\rho \nabla_\rho L_\nu}_{(A)} + K^\rho \nabla_\mu \nabla_\rho L_\nu - \underbrace{\nabla_\mu L^\rho \nabla_\rho K_\nu}_{(B)} - L^\rho \nabla_\mu \nabla_\rho K_\nu \\ &\quad + \underbrace{\nabla_\nu K^\rho \nabla_\rho L_\mu}_{(C)} + K^\rho \nabla_\nu \nabla_\rho L_\mu - \underbrace{\nabla_\nu L^\rho \nabla_\rho K_\mu}_{(D)} - L^\rho \nabla_\nu \nabla_\rho K_\mu \ . \end{aligned} \tag{2.6}$$

Since K and L are both Killing vectors, we know that

$$\begin{aligned} \nabla_\rho K_\mu &= -\nabla_\mu K_\rho \ , \\ \nabla_\nu L^\rho &= -\nabla^\rho L_\nu \ , \end{aligned} \tag{2.7}$$

i.e. we can swap the indices when a single derivative acts on them, at the cost of a minus sign. Thus, we find that

$$(D) = \nabla_\nu L^\rho \nabla_\rho K_\mu = \nabla^\rho L_\nu \nabla_\mu K_\rho = \nabla_\rho L_\nu \nabla_\mu K^\rho = \nabla_\mu K^\rho \nabla_\rho L_\nu = (A) \ , \tag{2.8}$$

so (A) and (D) cancel out in (2.6). Similarly, (B) and (C) cancel out in (2.6). We are left with

$$\nabla_\mu M_\nu + \nabla_\nu M_\mu = \underbrace{K^\rho \nabla_\mu \nabla_\rho L_\nu}_{(E)} - \underbrace{L^\rho \nabla_\mu \nabla_\rho K_\nu}_{(F)} + \underbrace{K^\rho \nabla_\nu \nabla_\rho L_\mu}_{(G)} - \underbrace{L^\rho \nabla_\nu \nabla_\rho K_\mu}_{(H)} \ . \tag{2.9}$$

To deal with the remaining terms, our strategy will be to use the commutator identity

$$\nabla_\mu \nabla_\rho = [\nabla_\mu, \nabla_\rho] + \nabla_\rho \nabla_\mu \ , \tag{2.10}$$

in order to swap the order of derivatives on all terms appearing in (2.9). Once we do this, there will be a number of terms that will look like the Killing vector equation and will thus automatically vanish. So, let's begin:

$$\begin{aligned} (E) &= K^\rho \nabla_\mu \nabla_\rho L_\nu = K^\rho [\nabla_\mu, \nabla_\rho] L_\nu + K^\rho \nabla_\rho \nabla_\mu L_\nu \ , \\ (G) &= K^\rho \nabla_\nu \nabla_\rho L_\mu = K^\rho [\nabla_\nu, \nabla_\rho] L_\mu + K^\rho \nabla_\rho \nabla_\nu L_\mu \ , \\ (F) &= L^\rho \nabla_\mu \nabla_\rho K_\nu = L^\rho [\nabla_\mu, \nabla_\rho] K_\nu L^\rho \nabla_\rho \nabla_\mu K_\nu \ , \\ (H) &= L^\rho \nabla_\nu \nabla_\rho K_\mu = L^\rho [\nabla_\nu, \nabla_\rho] K_\mu L^\rho \nabla_\rho \nabla_\nu K_\mu \ . \end{aligned} \tag{2.11}$$

Clearly, $(E) + (G)$ will have a term proportional to $\nabla_\mu L_\nu + \nabla_\nu L_\mu$, which vanishes. Similarly, a term like $\nabla_\mu K_\nu + \nabla_\nu K_\mu$ will vanish inside of $(F) + (H)$. Therefore, (2.9) becomes:

$$\nabla_\mu M_\nu + \nabla_\nu M_\mu = K^\rho([\nabla_\mu, \nabla_\rho]L_\nu + [\nabla_\nu, \nabla_\rho]L_\mu) - L^\rho([\nabla_\mu, \nabla_\rho]K_\nu + [\nabla_\nu, \nabla_\rho]K_\mu) \ . \tag{2.12}$$

Now, using the fact that the commutator acting on a vector is $[\nabla_\rho, \nabla_\sigma]V_\mu = -R^\nu{}_{\mu\rho\sigma}V_\nu$, we can write this as

$$\begin{aligned}
\nabla_\mu M_\nu + \nabla_\nu M_\mu &= -K^\rho(R^\lambda{}_{\nu\mu\rho} + R^\lambda{}_{\mu\nu\rho})L_\lambda + L^\rho(R^\lambda{}_{\nu\mu\rho} + R^\lambda{}_{\mu\nu\rho})K_\lambda \\
&= -K^\rho L^\lambda(R_{\lambda\nu\mu\rho} + R_{\lambda\mu\nu\rho}) + K^\lambda L^\rho(R_{\lambda\nu\mu\rho} + R_{\lambda\mu\nu\rho}) \\
&= K^\rho L^\lambda(-R_{\lambda\nu\mu\rho} - R_{\lambda\mu\nu\rho} + R_{\rho\nu\mu\lambda} + R_{\rho\mu\nu\lambda}) \\
&= K^\rho L^\lambda(-R_{\lambda\nu\mu\rho} - R_{\lambda\mu\nu\rho} + R_{\mu\lambda\rho\nu} + R_{\nu\lambda\rho\mu}) \\
&= K^\rho L^\lambda(-R_{\lambda\nu\mu\rho} - R_{\lambda\mu\nu\rho} + R_{\lambda\mu\nu\rho} + R_{\lambda\nu\mu\rho}) \\
&= 0 \ .
\end{aligned} \tag{2.13}$$

In words, what we did was relabel dummy indices so that we can pull out the same factor of $K^\rho L^\lambda$ on all terms, and then we used the symmetries of the Riemann tensor to show that everything ends up cancelling. Thus, we conclude that

$$\nabla_\mu M_\nu + \nabla_\nu M_\mu = 0 \ , \tag{2.14}$$

and thus $M = [K, L]$ is a Killing vector. Thus, the commutator of any two Killing vectors is another Killing vector.

7.) The covariant derivative ∇_μ satisfies the product rule, much in the same way that the ordinary derivative ∂_μ does. Therefore,

$$\nabla_\mu J^\mu = \nabla_\mu(T^{\mu\nu}K_\nu) = (\nabla_\mu T^{\mu\nu})K_\nu + T^{\mu\nu}(\nabla_\mu K_\nu) \ . \tag{2.15}$$

Since the stress-energy tensor is conserved, the first term on the far right-hand side of this expression is zero, and we are left with $\nabla_\mu J^\mu = T^{\mu\nu}\nabla_\mu K_\nu$. Then, since the stress-energy tensor is, *by definition*, symmetric, we can re-write this as

$$\nabla_\mu J^\mu = T^{\mu\nu}\nabla_\mu K_\nu = T^{\mu\nu}\nabla_{(\mu}K_{\nu)} = 0 \ , \tag{2.16}$$

where the last equality comes from the Killing vector equation $\nabla_{(\mu}K_{\nu)} = 0$.

8.) Using the chain rule, we know that

$$\begin{aligned}
\frac{d}{d\lambda}\left(K_\mu U^\mu\right) &= \frac{dx^\nu}{d\lambda}\nabla_\nu\left(K_\mu U^\mu\right) \\
&= U^\nu\nabla_\nu\left(K_\mu U^\mu\right) \\
&= U^\mu U^\nu\nabla_\nu K_\mu + K_\mu U^\nu\nabla_\nu U^\mu \ .
\end{aligned} \tag{2.17}$$

Since $U^\mu U^\nu$ is symmetric, the first term is equivalent to $U^\mu U^\nu \nabla_{(\nu K_\mu)}$, which vanishes by the Killing vector equation $\nabla_{(\mu}K_{\nu)} = 0$. The second term vanishes along a geodesic because the geodesic equation is simply $U^\nu\nabla_\nu U^\mu = 0$. Thus, both terms are zero and we are left with

$$K_\mu U^\mu \text{ is conserved along a geodesic.} \tag{2.18}$$

3 Riemann Normal Coordinates

9.) Remember, the Christoffel symbols are determined by the first derivative of the metric:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right) \ . \tag{3.1}$$

Since all of these derivatives vanish at P, we conclude that $\Gamma^{\rho}_{\mu\nu}(P) = 0$.

However, this does not mean that $\partial_\sigma \Gamma^{\rho}_{\mu\nu}(P)$ is zero! We only know that the first derivatives of the metric vanish at a single point; we know nothing about the second derivatives yet at that point. As is standard in calculus, a function can have its first derivative but *not* its second derivative vanish at a particular point.

10.) We can write out the first derivative of the Christoffel symbol more explicitly:

$$\begin{aligned} \partial_\sigma \Gamma^{\rho}_{\mu\nu} &= \partial_\sigma \left(\frac{1}{2} g^{\rho\lambda} \left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} \right) \right) \\ &= \frac{1}{2} (\partial_\sigma g^{\rho\lambda}) \left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} \right) \\ &\quad + \frac{1}{2} g^{\rho\lambda} \left(\partial_\sigma \partial_\mu g_{\nu\lambda} + \partial_\sigma \partial_\nu g_{\mu\lambda} - \partial_\sigma \partial_\lambda g_{\mu\nu} \right) \ . \end{aligned} \tag{3.2}$$

If we evaluate this at the point P, all first derivatives of the metric vanish, and so this simplifies to

$$\partial_\sigma \Gamma^{\rho}_{\mu\nu}(P) = \frac{1}{2} \eta^{\rho\lambda} \left(\partial_\sigma \partial_\mu g_{\nu\lambda}(P) + \partial_\sigma \partial_\nu g_{\mu\lambda}(P) - \partial_\sigma \partial_\lambda g_{\mu\nu}(P) \right) \ . \tag{3.3}$$

Now, recall that the Riemann tensor is in general given by

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho} + \Gamma^{\mu}_{\rho\lambda} \Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\mu}_{\sigma\lambda} \Gamma^{\lambda}_{\nu\rho} \ . \tag{3.4}$$

We have shown that $\Gamma^{\mu}_{\nu\rho}(P) = 0$, and we know from equation (3.3) what derivatives of the Christoffel symbol are, and so this whole equation taken at the point P becomes

$$\begin{aligned} R^{\mu}{}_{\nu\rho\sigma}(P) &= \partial_\rho \Gamma^{\mu}_{\nu\sigma}(P) - \partial_\sigma \Gamma^{\mu}_{\nu\rho}(P) \\ &= \frac{1}{2} \eta^{\mu\lambda} \left(\partial_\rho \partial_\nu g_{\sigma\lambda}(P) + \underbrace{\partial_\rho \partial_\sigma g_{\nu\lambda}(P)}_{\text{(cancels)}} - \partial_\rho \partial_\lambda g_{\nu\sigma}(P) \right) \\ &\quad - \frac{1}{2} \eta^{\mu\lambda} \left(\partial_\sigma \partial_\nu g_{\rho\lambda}(P) + \underbrace{\partial_\sigma \partial_\rho g_{\nu\lambda}(P)}_{\text{(cancels)}} - \partial_\sigma \partial_\lambda g_{\nu\rho}(P) \right) \\ &= \frac{1}{2} \eta^{\mu\lambda} \left(\partial_\rho \partial_\nu g_{\sigma\lambda}(P) - \partial_\rho \partial_\lambda g_{\nu\sigma}(P) - \partial_\sigma \partial_\nu g_{\rho\lambda}(P) + \partial_\sigma \partial_\lambda g_{\nu\rho}(P) \right) \ . \end{aligned} \tag{3.5}$$

Note that we used the fact that partial derivatives commute to cancel out the two indicated terms. From here, we can immediately see that the corresponding expression where all indices are lowered is

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= \frac{1}{2} \left(\partial_\rho \partial_\nu g_{\sigma\mu}(P) - \partial_\rho \partial_\mu g_{\nu\sigma}(P) - \partial_\sigma \partial_\nu g_{\rho\mu}(P) + \partial_\sigma \partial_\mu g_{\nu\rho}(P) \right) \\ &= \frac{1}{2} \left(\partial_\sigma \partial_\mu g_{\nu\rho}(P) - \partial_\rho \partial_\mu g_{\nu\sigma}(P) + \partial_\rho \partial_\nu g_{\mu\sigma}(P) - \partial_\sigma \partial_\nu g_{\mu\rho}(P) \right) \ , \end{aligned} \tag{3.6}$$

where in the last equality we moved terms around and used the fact that the metric is symmetric. This is the result we wanted to prove.

11.) **Bonus:** We will prove the Bianchi identity $\nabla_{[\lambda}R_{\mu\nu]\rho\sigma} = 0$ by first proving that, in our Riemann normal coordinate frame, $\nabla_{[\lambda}R_{\mu\nu]\rho\sigma}(P) = 0$, and then arguing that this result generalizes.

First, we note that since the Christoffel symbols vanish at the point P in Riemann normal coordinates, the covariant derivative acting on the Riemann tensor reduces simply to an ordinary derivative, and thus

$$\nabla_\lambda R_{\mu\nu\rho\sigma}(P) = \partial_\lambda R_{\mu\nu\rho\sigma}(P) \ . \tag{3.7}$$

Now, if we plug in the result (3.6) from problem 10, we can write the Riemann tensor in terms of derivatives acting on the metric, leaving us with

$$\nabla_\lambda R_{\mu\nu\rho\sigma}(P) = \frac{1}{2}\left(\partial_\lambda\partial_\sigma\partial_\mu g_{\nu\rho} - \partial_\lambda\partial_\rho\partial_\mu g_{\nu\sigma} + \partial_\lambda\partial_\rho\partial_\nu g_{\mu\sigma} - \partial_\lambda\partial_\sigma\partial_\nu g_{\mu\rho}\right)(P) \ , \tag{3.8}$$

where all terms in the right-hand side are evaluated at the point P. Similarly, rearranging the λ, μ, and ν indices in this expression also tells us that

$$\nabla_\mu R_{\nu\lambda\rho\sigma}(P) = \frac{1}{2}\left(\partial_\mu\partial_\sigma\partial_\nu g_{\lambda\rho} - \partial_\mu\partial_\rho\partial_\nu g_{\lambda\sigma} + \partial_\mu\partial_\rho\partial_\lambda g_{\nu\sigma} - \partial_\mu\partial_\sigma\partial_\lambda g_{\nu\rho}\right)(P) \ , \tag{3.9}$$

and

$$\nabla_\nu R_{\lambda\mu\rho\sigma}(P) = \frac{1}{2}\left(\partial_\nu\partial_\sigma\partial_\lambda g_{\mu\rho} - \partial_\nu\partial_\rho\partial_\lambda g_{\mu\sigma} + \partial_\nu\partial_\rho\partial_\mu g_{\lambda\sigma} - \partial_\nu\partial_\sigma\partial_\mu g_{\lambda\rho}\right)(P) \ . \tag{3.10}$$

My claim now is that if we add up (3.8), (3.8), and (3.10) and use fact that partial derivatives commute ($\partial_\alpha\partial_\beta = \partial_\beta\partial_\alpha$), then the result is zero. I have highlighted the individual terms that cancel in the same colors as one another, hopefully for clarity. You should go through and agree that everything does cancel against one another, leaving us with:

$$\nabla_\lambda R_{\mu\nu\rho\sigma}(P) + \nabla_\mu R_{\nu\lambda\rho\sigma}(P) + \nabla_\nu R_{\lambda\mu\rho\sigma}(P) = 0 \ . \tag{3.11}$$

With this in mind, we can now prove that the Bianchi identity holds at the point P. Remember, the bracket notation $[\lambda\mu\nu]$ means the fully antisymmetric combination of those three indices, weighted overall by a factor of $1/3!$. That is, the definition is such that

$$\begin{aligned}\nabla_{[\lambda}R_{\mu\nu]\rho\sigma} \equiv \frac{1}{3!}(&\nabla_\lambda R_{\mu\nu\rho\sigma} + \nabla_\mu R_{\nu\lambda\rho\sigma} + \nabla_\nu R_{\lambda\mu\rho\sigma} \\ &- \nabla_\lambda R_{\nu\mu\rho\sigma} - \nabla_\mu R_{\lambda\nu\rho\sigma} - \nabla_\nu R_{\mu\lambda\rho\sigma}) \ .\end{aligned} \tag{3.12}$$

Since the Riemann tensor is antisymmetric in its first pair of indices, the second line turns out to actually just be identical to the first line, and thus we are left with

$$\nabla_{[\lambda}R_{\mu\nu]\rho\sigma} = \frac{1}{3}(\nabla_\lambda R_{\mu\nu\rho\sigma} + \nabla_\mu R_{\nu\lambda\rho\sigma} + \nabla_\nu R_{\lambda\mu\rho\sigma}) \ . \tag{3.13}$$

Combining this with our result from (3.11) thus immediately tells us that

$$\nabla_{[\lambda}R_{\mu\nu]\rho\sigma}(P) = 0 \ . \tag{3.14}$$

So, the Bianchi identity is true at the point P in our Riemann normal coordinate frame. But, since this is a tensor equation, it holds irrespective of the coordinate frame we are in. Additionally, since there was nothing special about the point P we picked for our Riemann normal coordinate frame, the same result should hold at all points on our spacetime manifold. Hence the Bianchi identity will also be true at all points in any arbitrary coordinate frame.

12.) Since the Christoffel symbols vanish at P, the geodesic equation at the point P simply becomes

$$\frac{d^2x^\mu}{d\lambda^2} = 0 \ . \tag{3.15}$$

This is in general solved by $x^\mu(\lambda) = c^\mu\lambda + d^\mu$, for some constants c^μ and d^μ. Without loss of generality, we can choose the point P (which is chosen to be the origin of our spacetime) to correspond to $\lambda = 0$, which in turn sets $d^\mu = 0$. Thus, $x^\mu(\lambda) = c^\mu\lambda$ near P for any geodesic that goes through the point P. If we were to solve this far from P, we would need to know the Christoffel symbols away from P, but locally this suffices.

13.) Differentiating the geodesic equation and applying the chain rule tells us that

$$\begin{aligned} 0 &= \frac{d}{d\lambda}\left(\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda}\right) \\ &= \frac{d^3x^\mu}{d\lambda^3} + \frac{d}{d\lambda}\left(\Gamma^\mu_{\rho\sigma}\right)\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} + \Gamma^\mu_{\rho\sigma}\left(\frac{d^2x^\rho}{d\lambda^2}\frac{dx^\sigma}{d\lambda} + \frac{dx^\rho}{d\lambda}\frac{d^2x^\sigma}{d\lambda^2}\right) \\ &= \frac{d^3x^\mu}{d\lambda^3} + \frac{dx^\nu}{d\lambda}\left(\partial_\nu\Gamma^\mu_{\rho\sigma}\right)\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} + \Gamma^\mu_{\rho\sigma}\left(\frac{d^2x^\rho}{d\lambda^2}\frac{dx^\sigma}{d\lambda} + \frac{dx^\rho}{d\lambda}\frac{d^2x^\sigma}{d\lambda^2}\right) \ . \end{aligned} \tag{3.16}$$

At the point P, we know that geodesics take the form $x^\mu(\lambda) = c^\mu\lambda$, so their second and third derivatives with respect to λ vanish, and additionally that $\Gamma^\mu_{\rho\sigma}(P) = 0$. Thus, this equation at the point P simply reduces to

$$0 = (\partial_\nu\Gamma^\mu_{\rho\sigma}(P))c^\nu c^\rho c^\sigma \ . \tag{3.17}$$

Since $c^\nu c^\rho c^\sigma$ is fully symmetric in the three indices, this can also be written as

$$0 = (\partial_{(\nu}\Gamma^\mu_{\rho\sigma)}(P))c^\nu c^\rho c^\sigma \ . \tag{3.18}$$

This equation holds true for arbitrary constants c^μ, since for any such set of constants we can construct the corresponding geodesic. Thus, the only way to solve this is to demand that $\partial_{(\nu}\Gamma^\mu_{\rho\sigma)}(P) = 0$. Lastly, the Christoffel symbols are symmetric in their lower indices, and thus this is equivalent to demanding that

$$\partial_\nu\Gamma^\mu_{\rho\sigma}(P) + \partial_\rho\Gamma^\mu_{\sigma\nu}(P) + \partial_\sigma\Gamma^\mu_{\nu\rho}(P) = 0 \ . \tag{3.19}$$

14.) **Bonus:** Consider taking n derivatives of the geodesic equation with respect to λ:

$$0 = \frac{d^{n+2}x^\mu}{d\lambda^{n+2}} + \frac{d^n}{d\lambda^n}\left(\Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda}\right) \ . \tag{3.20}$$

Near the point P, we know that $x^\mu(\lambda) = c^\mu\lambda$, and so the first term clearly vanishes, since taking $n+2$ derivatives on a single instance of λ will yield zero. The remaining second term in the geodesic equation becomes (using the chain rule):

$$\begin{aligned} 0 &= \frac{d^n}{d\lambda^n}\left(\Gamma_{\rho\sigma}(P)^\mu c^\rho c^\lambda\right) \\ &= \frac{dx^{\nu_1}}{d\lambda}\partial_{\nu_1}\left(\frac{dx^{\nu_2}}{d\lambda}\partial_{\nu_2}\left(\ldots\frac{dx^{\nu_n}}{d\lambda}\partial_{\nu_n}\left(\Gamma^\mu_{\rho\sigma}(P)c^\rho c^\sigma\right)\ldots\right)\right) \\ &= \partial_{\nu_1}\partial_{\nu_2}\ldots\partial_{\nu_n}\left(\Gamma^\mu_{\rho\sigma}(P)\right)c^{\nu_1}c^{\nu_2}\ldots c^{\nu_n}c^\rho c^\sigma \ . \end{aligned} \tag{3.21}$$

That is, the only term that survives is the one where all derivatives act on the Christoffel symbols, since the c^μ are constants. Then, we once again note that $c^{\nu_1}c^{\nu_2}\ldots c^{\nu_n}c^\rho c^\sigma$ is fully symmetric in all of its indices, and so we can write this as

$$\partial_{(\nu_1}\partial_{\nu_2}\ldots\partial_{\nu_n}\Gamma^\mu_{\rho\sigma)}(P) = 0 \ . \tag{3.22}$$

15.) First, recall from equation (3.5) that since all Christoffel symbols vanish at P, the Riemann tensor at P is simply

$$R^{\mu}{}_{\nu\rho\sigma}(P) = \partial_\rho \Gamma^\mu_{\nu\sigma}(P) - \partial_\sigma \Gamma^\mu_{\nu\rho}(P) \ . \tag{3.23}$$

From this, it immediately follows that

$$R^{\mu}{}_{\rho\nu\sigma}(P) + R^{\mu}{}_{\nu\rho\sigma}(P) = \partial_\rho \Gamma^\mu_{\nu\sigma}(P) + \partial_\nu \Gamma^\mu_{\rho\sigma}(P) - 2\partial_\sigma \Gamma^\mu_{\nu\rho}(P) \ . \tag{3.24}$$

Our result from problem 13 then tells us that, using the symmetry of the Christoffel symbols, the first two terms on the right-hand side of the above equation satisfy

$$\partial_\rho \Gamma^\mu_{\nu\sigma}(P) + \partial_\nu \Gamma^\mu_{\rho\sigma}(P) = -\partial_\sigma \Gamma^\mu_{\nu\rho}(P) \ , \tag{3.25}$$

and thus we find that

$$R^{\mu}{}_{\rho\nu\sigma}(P) + R^{\mu}{}_{\nu\rho\sigma}(P) = -3\partial_\sigma \Gamma^\mu_{\nu\rho}(P) \ . \tag{3.26}$$

We can easily rewrite this as

$$\partial_\sigma \Gamma^\mu_{\nu\rho}(P) = -\frac{1}{3}\left(R^{\mu}{}_{\rho\nu\sigma}(P) + R^{\mu}{}_{\nu\rho\sigma}(P)\right) \ . \tag{3.27}$$

This proves the intermediate step listed in the problem.

Armed with this, let's explicitly write out $\partial_\sigma \Gamma^\mu_{\nu\rho}(P)$ in terms of derivatives of the metric, keeping in mind that first derivatives vanish:

$$\partial_\sigma \Gamma^\lambda_{\nu\rho}(P) = \frac{1}{2}\eta^{\lambda\mu}\left(\partial_\sigma\partial_\nu g_{\rho\mu}(P) + \partial_\sigma\partial_\rho g_{\nu\mu}(P) - \partial_\sigma\partial_\mu g_{\nu\rho}(P)\right) \ , \tag{3.28}$$

which in turn means that

$$\eta_{\mu\lambda}\partial_\sigma \Gamma^\lambda_{\nu\rho}(P) = \frac{1}{2}\left(\partial_\sigma\partial_\nu g_{\mu\rho}(P) + \partial_\sigma\partial_\rho g_{\mu\nu}(P) - \partial_\sigma\partial_\mu g_{\nu\rho}(P)\right) \ . \tag{3.29}$$

Notice that the last two terms are antisymmetric under $\mu \leftrightarrow \rho$, and thus we can add the symmetric part to cancel them and solve for $\partial_\sigma\partial_\nu g_{\mu\rho}(P)$:

$$\partial_\sigma\partial_\nu g_{\mu\rho}(P) = \eta_{\mu\lambda}\partial_\sigma \Gamma^\lambda_{\nu\rho}(P) + \eta_{\rho\lambda}\partial_\sigma \Gamma^\lambda_{\nu\mu}(P) \ . \tag{3.30}$$

Lastly, if we use the intermediate result (3.27) to express the right-hand side of this in terms of the Riemann tensor, we find that

$$\begin{aligned}
\partial_\sigma\partial_\nu g_{\mu\rho}(P) &= -\frac{1}{3}\eta_{\mu\lambda}\left(R^{\lambda}{}_{\rho\nu\sigma}(P) + R^{\lambda}{}_{\nu\rho\sigma}(P)\right) - \frac{1}{3}\eta_{\rho\lambda}\left(R^{\lambda}{}_{\mu\nu\sigma}(P) + R^{\lambda}{}_{\nu\mu\sigma}(P)\right) \\
&= -\frac{1}{3}\left(\underbrace{R_{\mu\rho\nu\sigma}(P)}_{\text{cancels}} + R_{\mu\nu\rho\sigma}(P) + \underbrace{R_{\rho\mu\nu\sigma}(P)}_{\text{cancels}} + R_{\rho\nu\mu\sigma}(P)\right) \\
&= -\frac{1}{3}\left(R_{\mu\nu\rho\sigma}(P) + R_{\rho\nu\mu\sigma}(P)\right) \ .
\end{aligned} \tag{3.31}$$

Rearranging the indices slightly, and using commutativity of partial derivatives, we find the desired result:

$$\partial_\rho\partial_\sigma g_{\mu\nu}(P) = -\frac{1}{3}\left(R_{\mu\rho\nu\sigma}(P) + R_{\mu\sigma\nu\rho}(P)\right) \ . \tag{3.32}$$

16.) The metric at the point P', expressed in terms of a Taylor expansion around the point P, is in general exactly what you would expect:

$$g_{\mu\nu}(P') = g_{\mu\nu}(P) + \partial_\rho g_{\mu\nu}(P) x^\rho + \frac{1}{2!}\partial_\rho\partial_\sigma g_{\mu\nu}(P) x^\rho x^\sigma + \frac{1}{3!}\partial_\rho\partial_\sigma\partial_\lambda g_{\mu\nu}(P) x^\rho x^\sigma x^\lambda + \ldots \tag{3.33}$$

Note that we used the fact that P' is at coordinate x^μ and P at the origin, so the coordinate distance between the points really is just x^μ. From here on out, assume x^μ is small so we can neglect the terms that are $\mathcal{O}(x^3)$ or higher. Then, plug in $g_{\mu\nu}(P) = \eta_{\mu\nu}$, $\partial_\rho g_{\mu\nu}(P) = 0$, and the result from the previous problem:

$$g_{\mu\nu}(P') = \eta_{\mu\nu} - \frac{1}{6}(R_{\mu\rho\nu\sigma}(P) + R_{\mu\sigma\nu\rho}(P)) x^\rho x^\sigma + \mathcal{O}(x^3) \ . \tag{3.34}$$

Since $x^\rho x^\sigma$ is symmetric under $\rho \leftrightarrow \sigma$, we can immediately see that $R_{\mu\rho\nu\sigma} x^\rho x^\sigma = R_{\mu\sigma\nu\rho} x^\rho x^\sigma$, i.e. when contracted with $x^\rho x^\sigma$ we can simply take the part of $R_{\mu\rho\nu\sigma}$ that is symmetric under $\rho \leftrightarrow \sigma$. Thus, we are finally left with

$$g_{\mu\nu}(P') = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\rho\nu\sigma}(P) x^\rho x^\sigma + \mathcal{O}(x^3) \ . \tag{3.35}$$